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Residuated mapping and CPS-translation
– Extended abstract –

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Abstract

We provide a call-by-name CPS-translation from polymorphic λ-calculus $\lambda 2$ into existential λ-calculus $\lambda^{\exists}$. Then we prove that the CPS-translation is a residuated mapping from the preordered set of $\lambda 2$-terms to that of $\lambda^{\exists}$-terms. From the inductive proof, its residual (inverse translation) can be extracted, which constitutes the so-called Galois connection. It is also obtained that given the CPS-translation the existence of its inverse is unique.

1 Preliminaries

By a preordered set $\langle A, \sqsubseteq \rangle$, we mean a set A on which there is defined a preorder, i.e., a reflexive and transitive relation $\sqsubseteq$. If $\langle A_1, \sqsubseteq_1 \rangle$ and $\langle A_2, \sqsubseteq_2 \rangle$ are preordered sets, then we say that a mapping $f : A_1 \to A_2$ is monotone, if $x \sqsubseteq_1 y$ implies $f(x) \sqsubseteq_2 f(y)$ for any $x, y \in A_1$. A direct image under f is denoted by $f[X]$ for every $X \subseteq A_1$, and an inverse image is denoted by $f^{\leftarrow}[Y]$ for every $Y \subseteq A_2$. A subset $B \subseteq A$ is a down-set of a preodered set $\langle A, \sqsubseteq \rangle$, if $y \sqsubseteq x$ together with $y \in A$ and $x \in B$ implies $y \in B$. By a principal down-set, we mean a down-set of the form $\{y \in A \mid y \sqsubseteq x\}$, which is denoted by $\downarrow x$.

Definition 1 (Residuated mapping) *A mapping $f : A \to B$ that satisfies the following condition is said to be residuated: The inverse image under f of every principal down-set of B is a principal down-set of A.*

2 Source calculus: $\lambda 2$

We introduce our source calculus of 2nd order λ-calculus (Girard-Reynolds), denoted by $\lambda 2$. For simplicity, we adopt its domain-free style.

Definition 2 (Types)

$$A ::= X \mid A \Rightarrow A \mid \forall X.A$$

Definition 3 ((Pseudo)λ2-terms)

$$\Lambda 2 \ni M ::= x \mid \lambda x.M \mid MM \mid \lambda X.M \mid MA$$

Definition 4 (Reduction rules) (β) $(\lambda x.M_1)M_2 \rightarrow M_1[x := M_2]$

(η) $\lambda x.Mx \rightarrow M,$ *if* $x \notin FV(M)$

(β_t) $(\lambda X.M)A \rightarrow M[X := A]$

(η_t) $\lambda X.MX \rightarrow M,$ *if* $X \notin FV(M)$

$FV(M)$ denotes a set of free variables in M.

We write $\twoheadrightarrow_{\lambda 2}$ for the compatible relation obtained from the reflexivie and transitive closure of the one step reduction relation, and $\rightarrow^+_{\lambda 2}$ for that from the transitive closure. In particular, $\twoheadrightarrow_R$ denotes the subrelation of $\twoheadrightarrow$ restricted to the reduction rules $R \subseteq \{\beta, \eta, \beta_t, \eta_t\}$. We may write simply (β) for either (β) or (β_t), and (η) for either (η) or (η_t), if clear from the context. We employ the notation $\equiv$ to indicate the syntactic identity under renaming of bound variables.

3 Target calculus: $\lambda^{\exists}$

We next define our target calculus denoted by $\lambda^{\exists}$, which is logically a subsystem of minimal logic consisting of constant $\bot$, negation, conjunction and 2nd order existential quantification[1].

Definition 5 (Types)

$$A ::= \bot \mid X \mid \neg A \mid A \wedge A \mid \exists X.A$$

Definition 6 ((Pseudo)$\lambda^{\exists}$-terms)

$$\begin{aligned} \Lambda^{\exists} \ni M \quad ::= \quad & x \mid \lambda x.M \mid MM \mid \langle M, M\rangle \mid \texttt{let}\ \langle x, x\rangle = M\ \texttt{in}\ M \\ & \mid \langle A, M\rangle \mid \texttt{let}\ \langle X, x\rangle = M\ \texttt{in}\ M \end{aligned}$$

Definition 7 (Reduction rules) (β) $(\lambda x.M_1)M_2 \rightarrow M_1[x := M_2]$

(η) $\lambda x.Mx \rightarrow M,$ *if* $x \notin FV(M)$

$(\texttt{let}_{\wedge})$ $\texttt{let}\ \langle x_1, x_2\rangle = \langle M_1, M_2\rangle\ \texttt{in}\ M \rightarrow M[x_1 := M_1, x_2 := M_2]$

$(\texttt{let}_{\wedge_\eta})$ $\texttt{let}\ \langle x_1, x_2\rangle = M_1\ \texttt{in}\ M[z := \langle x_1, x_2\rangle] \rightarrow M[z := M_1],$ *if* $x_1, x_2 \notin FV(M)$

$(\texttt{let}_{\exists})$ $\texttt{let}\ \langle X, x\rangle = \langle A, M_1\rangle\ \texttt{in}\ M \rightarrow M[X := A, x := M_1]$

$(\texttt{let}_{\exists_\eta})$ $\texttt{let}\ \langle X, x\rangle = M_1\ \texttt{in}\ M[z := \langle X, x\rangle] \rightarrow M_2[z := M_1],$ *if* $X, x \notin FV(M_2)$

We also write simply $(\texttt{let})$ for either $(\texttt{let}_{\wedge})$ or $(\texttt{let}_{\exists})$, and $(\texttt{let}_{\eta})$ for $(\texttt{let}_{\wedge_\eta})$ or $(\texttt{let}_{\exists_\eta})$. Similarly we write $\twoheadrightarrow_{\lambda^{\exists}}$ and $\rightarrow^+_{\lambda^{\exists}}$ as done for $\lambda 2$.

[1] For further introduction of the CPS target calculus $\lambda^{\exists}$ with let-expressions, see also [5].

4 CPS-translation * from $\Lambda 2$ into $\Lambda^{\exists}$

We define a translation, so-called modified CPS-translation * from pseudo $\lambda 2$-terms into pseudo $\lambda^{\exists}$-terms. In each case, a fresh and free variable a is introduced, which is called a continuation variable.

Definition 8 *1.* $x^* = xa$

2. $(\lambda x.M)^* = \mathtt{let}\ \langle x, a\rangle = a\ \mathtt{in}\ M^*$

3. $(M_1M_2)^* = \begin{cases} M_1^*[a := \langle x, a\rangle] & \textit{for } M_2 \equiv x \\ M_1^*[a := \langle \lambda a.M_2^*, a\rangle] & \textit{otherwise} \end{cases}$

4. $(\lambda X.M)^* = \mathtt{let}\ \langle X, a\rangle = a\ \mathtt{in}\ M^*$

5. $(MA)^* = M^*[a := \langle A^*, a\rangle]$

6. $X^* = X;\quad (A_1 \Rightarrow A_2)^* = \neg A_1^* \wedge A_2^*;\quad (\forall X.A)^* = \exists X.A^*$

Remarked that M^* contains exactly one free occurrence of a continuation variable a, and M^* has neither β-redex nor η-redex. Let $\lambda X.M$ have type $\forall X.A$. Then, under the translation, the parametric polymorphic function $\lambda X.M$ with respect to X becomes an abstract data type $(\lambda X.M)^*$ for X, which is waiting for an implementation a with type $\exists X.A^*$ together with an interface (a signature) with type A^*, i.e., $(\lambda X.M)^*$ is

$$\mathtt{abstype}\ X\ \mathtt{with}\ a{:}A^*\ \mathtt{is}\ a\ \mathtt{in}\ M^*$$

in a familiar notation.

Lemma 1 (Monotone *) *If we have* $M_1 \rightarrow_{\lambda 2} M_2$*, then* $M_1^* \rightarrow^+_{\lambda\exists} M_2^*$ *holds. In particular, if* $M_1 \rightarrow_\beta M_2$*, then* $M_1^* \rightarrow^+_{\beta 1\mathtt{et}} M_2^*$*. And if* $M_1 \rightarrow_\eta M_2$*, then* $M_1^* \rightarrow_{1\mathtt{et}_\eta} M_2^*$*.*

Proof. By induction on the derivation. □

In order to give an inverse translation, first we provide the mutual inductive definitions, respectively for denotations *Univ* and continuations $\mathcal{C}$, as follows. Both *Univ* and $\mathcal{C}$ are down-sets in the above sense.

$$a \in \mathcal{C} \qquad\qquad \frac{C \in \mathcal{C}}{\langle x, C\rangle \in \mathcal{C}}$$

$$\frac{C \in \mathcal{C} \quad P \in \mathit{Univ}}{\langle \lambda a.P, C\rangle \in \mathcal{C}} \qquad\qquad \frac{C \in \mathcal{C}}{\langle A^*, C\rangle \in \mathcal{C}}$$

$$\frac{C \in \mathcal{C}}{xC \in \mathit{Univ}} \qquad\qquad \frac{C \in \mathcal{C} \quad P \in \mathit{Univ}}{(\lambda a.P)C \in \mathit{Univ}}$$

$$\frac{C \in \mathcal{C} \quad P \in \mathit{Univ}}{\mathtt{let}\ \langle x, a\rangle = C\ \mathtt{in}\ P \in \mathit{Univ}} \qquad\qquad \frac{C \in \mathcal{C} \quad P \in \mathit{Univ}}{\mathtt{let}\ \langle X, a\rangle = C\ \mathtt{in}\ P \in \mathit{Univ}}$$

We write $\langle R_1, R_2, \ldots, R_n\rangle$ for $\langle R_1, \langle R_2, \ldots, R_n\rangle\rangle$ with $n > 1$, and $\langle R_1\rangle$ for R_1 with $n = 1$. $C \in \mathcal{C}$ is in the form of $\langle R_1, \ldots, R_n, a\rangle$ where R_i $(1 \leq i \leq n)$ is x, $\lambda a.P$, or A^* with $n \geq 0$. We explicitly mention that $C \in \mathcal{C}$ has exactly one occurence of free variable a such that $C \equiv \langle R_1, \ldots, R_n, a\rangle$ with $n \geq 0$. $P \in$ *Univ* also has exactly one occurence of free variable a in such C as a proper subterm of P.

The inductively defined sets *Univ*, $\mathcal{C} \subseteq \Lambda^{\exists}$ are down-sets with respect to $\twoheadrightarrow_{\lambda^{\exists}}$.

Lemma 2 *1. If $P_1 \in$ Univ and $P_1 \twoheadrightarrow_{\lambda^{\exists}} P_2$, then $P_2 \in$ Univ.*

2. If $C_1 \in \mathcal{C}$ and $C_1 \twoheadrightarrow_{\lambda^{\exists}} C_2$, then $C_2 \in \mathcal{C}$.

Proof. Let $P, P_1 \in$ *Univ* and $C, C_1 \in \mathcal{C}$. Then $P[a := C_1], P[x := \lambda a.P_1], P[X := A^*] \in$ *Univ*, and $C[a := C_1], C[x := \lambda a.P_1], C[X := A^*] \in \mathcal{C}$. □

Proposition 1 ***1. Univ is strongly normalizing with respect to $\rightarrow_{\beta\eta}$, i.e., for any $P \in$ Univ, there is no infinite reduction sequence of $\rightarrow_{\beta\eta}$ starting with P.***

2. Univ is Church-Rosser with respect to $\twoheadrightarrow_{\beta\eta}$, i.e., for any $P, P_1, P_2 \in$ Univ, if we have $P \twoheadrightarrow_{\beta\eta} P_1$ and $P \twoheadrightarrow_{\beta\eta} P_2$, then there exists some $P_3 \in$ Univ such that $P_1 \twoheadrightarrow_{\beta\eta} P_3$ and $P_2 \twoheadrightarrow_{\beta\eta} P_3$.

Proof.

1. Since every λ-abstraction $\lambda a.P \in$ *Univ* is linear, for any $P_1 \rightarrow_{\beta\eta} P_2$, the contractum P_2 has less length than that of P_1.
2. *Univ* is weak Church-Rosser with respect to $\rightarrow_{\beta\eta}$, and hence the property of Church-Rosser holds from Newman's Lemma. □

Any (pseudo) term $P \in$ *Univ* is Church-Rosser and strongly normalizing with respect to $\beta\eta$-reductions, and the unique $\beta\eta$-normal form is denoted by $\Downarrow_{\beta\eta} P$. The same property naturally holds for $\mathcal{C}$ as well. A nomalization function $\Downarrow_{\beta\eta}$ can be inductively defined as follows:

Definition 9 ($\Downarrow_{\beta\eta}$) *1. For $P \in$ Univ:*

- *(a)* $\Downarrow_{\beta\eta}(xC) = x(\Downarrow_{\beta\eta} C)$
- *(b)* $\Downarrow_{\beta\eta}((\lambda a.P)C) = \Downarrow_{\beta\eta}(P[a := C])$
- *(c)* $\Downarrow_{\beta\eta}(\texttt{let } \langle \chi, a\rangle = C \texttt{ in } P) = \texttt{let } \langle \chi, a\rangle = \Downarrow_{\beta\eta} C \texttt{ in } \Downarrow_{\beta\eta} P$

2. For $C \equiv \langle R_1, \ldots, R_n, a\rangle \in \mathcal{C}$ with $n \geq 0$, where $R_i \equiv x, \lambda a.P$, or A^*:

$\Downarrow_{\beta\eta} \langle R_1, \ldots, R_n, a\rangle = \langle \Downarrow_{\beta\eta} R_1, \ldots, \Downarrow_{\beta\eta} R_n, a\rangle$

- *(a)* $R \equiv x$:

  $\Downarrow_{\beta\eta} x = x$
- *(b)* $R \equiv \lambda a.P$:
  - *i.* $\Downarrow_{\beta\eta}(\lambda a.xa) = x$, *if* $P \equiv xa$;
  - *ii.* $\Downarrow_{\beta\eta}(\lambda a.P) = \lambda a.(\Downarrow_{\beta\eta} P)$, *otherwise;*
- *(c)* $R \equiv A^*$:

  $\Downarrow_{\beta\eta} A^* = A^*$

5 Residuated CPS-translation

Proposition 2 *The following conditions are equivalent.*

1. $f : A \to B$ *is a residuated mapping.*

2. $f : A \to B$ *is monotone and there exists a monotone mapping* $g : B \to A$ *such that* $A \ni a \sqsubseteq g(f(a))$ *and* $f(g(b)) \sqsubseteq b \in B$.

Proof. A residuated mapping is monotone in general. On the other hand, from the condition 1, for any $b \in B$ there exists $a \in A$ such that $f^{\leftarrow}[\downarrow b] = \downarrow a$ which cannot be empty, whence one has a choice function $g : B \to A$ by $g(b) = a$. Hence $g(b) \in \downarrow g(b) = f^{\leftarrow}[\downarrow b]$ holds true, so that we have $f(g(b)) \sqsubseteq b$. We also have $a \in f^{\leftarrow}[\downarrow f(a)] = \downarrow g(f(a))$ by the definition, and hence we have $a \sqsubseteq g(f(a))$.

From the condition 2, we have that $f(a) \sqsubseteq b$ if and only if $a \sqsubseteq g(b)$. Hence, we have $f^{\leftarrow}[\downarrow b] = \downarrow g(b)$ for every $b \in B$. □

We write $M \sqsubseteq N$ for $N \twoheadrightarrow M$, i.e., the contextual and reflexive-transitive closure of one-step reduction $\to$.

Lemma 3 *For any* $P \in$ *Univ, there uniquely exists* $M \in \Lambda 2$ *such that* $\Downarrow_{\beta\eta} P \equiv M^*$.

Proof. By induction on $P \in$ *Univ*.

1. Case of $P \equiv xC \equiv x\langle R_1, \ldots, R_n, a\rangle$ with $n \geq 0$

   (a) If $R_i \equiv x_i$, then we take $N_i \equiv x_i$, whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.

   (b) Case of $R_i \equiv \lambda a.P_i$
   If $P_i \equiv x_i a$, then we take $N_i \equiv x_i$, and whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.
   Otherwise, from the induction hypothesis for P_i, there uniquely exists N_i such that $\Downarrow_{\beta\eta} P_i \equiv N_i^*$. Now we have $\Downarrow_{\beta\eta} R_i = \lambda a.(\Downarrow_{\beta\eta} P_i) \equiv \lambda a.N_i^*$.

   (c) If $R_i \equiv A_i^*$, then we take $N_i \equiv A_i$.

   Hence, we take $M \equiv xN_1 \ldots N_n$, and then there uniquely exists $M \in \Lambda 2$ such that
   $\Downarrow_{\beta\eta} P$
   $= x\langle \Downarrow_{\beta\eta} R_1, \ldots, \Downarrow_{\beta\eta} R_n, a\rangle$
   $\equiv x\langle N_1^{*\prime}, \ldots, N_n^{*\prime}, a\rangle$
   $= M^*$,
   where $N_i^{*\prime} = \lambda a.N_i^*$ if $R_i \equiv \lambda a.P_i$ with no outmost η-redex; otherwise $N_i^{*\prime} = N_i^*$.

2. Case of $P \equiv (\lambda a.P')C$
   Since a is a linear variable, by the induction hypothesis for $P'[a := C]$, there uniquely exists $M \in \Lambda 2$ such that $\Downarrow_{\beta\eta} (P'[a := C]) \equiv M^*$. Therefore, we have a unique $M \in \Lambda 2$ such that $\Downarrow_{\beta\eta} P \equiv M^*$.

3. Case of $P \equiv \texttt{let}\ \langle x, a\rangle = C\ \texttt{in}\ P_1$ with $C = \langle R_1, \ldots, R_n, a\rangle$ and $n \geq 0$

(a) From the induction hypothesis for P_1, there uniquely exists $M_1 \in \Lambda 2$ such that $\Downarrow_{\beta\eta} P_1 \equiv M_1^*$.

(b) If $R_i \equiv x_i$, then we take $N_i \equiv x_i$, whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.

(c) Case of $R_i \equiv \lambda a.P_i$
If $P_i \equiv x_i a$, then we take $N_i \equiv x_i$, and whence $\Downarrow_{\beta\eta} R_i \equiv x_i \equiv N_i^*$.
Otherwise, from the induction hypothesis for P_i, there uniquely exists N_i such that $\Downarrow_{\beta\eta} P_i \equiv N_i^*$. Now we have $\Downarrow_{\beta\eta} R_i = \lambda a.(\Downarrow_{\beta\eta} P_i) \equiv \lambda a.N_i^*$.

(d) If $R_i \equiv A_i^*$, then we take $N_i \equiv A_i$.

Hence, we take $M \equiv xN_1 \dots N_n$, and then there uniquely exists $M \in \Lambda 2$ such that

$$\begin{aligned}&\Downarrow_{\beta\eta} P\\ &= \texttt{let}\ \langle x, a\rangle = \langle \Downarrow_{\beta\eta} R_1, \dots, \Downarrow_{\beta\eta} R_n, a\rangle\ \texttt{in}\ (\Downarrow_{\beta\eta} P_1)\\ &\equiv \texttt{let}\ \langle x, a\rangle = \langle N_i^{*\prime}, \dots, N_i^{*\prime}, a\rangle\ \texttt{in}\ M_1^*\\ &= M^*,\end{aligned}$$

where $N_i^{*\prime} = \lambda a.N_i^*$ if $R_i \equiv \lambda a.P_i$ with no outmost η-redex; otherwise $N_i^{*\prime} = N_i^*$.

4. Case of $P \equiv \texttt{let}\ \langle X, a\rangle = C\ \texttt{in}\ P'$ is handled simiarly. □

From the inductive proof of Lemma 3 above, an extracted function giving a witness is written down here.

1. $x^\sharp = x$; $(\lambda a.P)^\sharp = P^\sharp$; $(A^*)^\sharp = A$
2. $(x\langle R_1, \dots, R_n, a\rangle)^\sharp = xR_1^\sharp \dots R_n^\sharp$
3. $((\lambda a.P)C)^\sharp = (P[a := C])^\sharp$
4. $(\texttt{let}\ \langle x, a\rangle = \langle R_1, \dots, R_n, a\rangle\ \texttt{in}\ P)^\sharp = (\lambda x.P^\sharp)R_1^\sharp \dots R_n^\sharp$
5. $(\texttt{let}\ \langle X, a\rangle = \langle R_1, \dots, R_n, a\rangle\ \texttt{in}\ P)^\sharp = (\lambda X.P^\sharp)R_1^\sharp \dots R_n^\sharp$

where the clause 1 is for R_i appeared in $\langle R_1, \dots, R_n, a\rangle \in C$, and the clause 2 through 5 are for $P \in Univ$.

Corollary 1 (Composition of * and $\sharp$) *1. For any $P \in Univ$, we have $P \twoheadrightarrow_{\beta\eta} (P^\sharp)^*$.*

2. For any $M \in \Lambda 2$, we have $(M^*)^\sharp \equiv M$.

Proof.

1. From Lemma 3, we have $\Downarrow_{\beta\eta} P \equiv (P^\sharp)^*$ and $P \twoheadrightarrow_{\beta\eta} \Downarrow_{\beta\eta} P$. Therefore, $P \twoheadrightarrow_{\beta\eta} (P^\sharp)^*$ holds for any $P \in Univ$.

2. From the definition of *, M^* has neither β- nor η-redex. Hence, $\Downarrow_{\beta\eta} (M^*) \equiv M^*$ holds, and then $(M^*)^\sharp \equiv M$ for any $M \in \Lambda 2$. □

Lemma 4 (Monotone $\sharp$) *The above mapping $\sharp : Univ \to \Lambda 2$ is monotone.*

Proof. By the definition of $\sharp$. In particular, let $P_1, P_2 \in \mathit{Univ}$, then the following holds.

1. If $P_1 \to_{\beta\eta} P_2$, then $P_1^\sharp \equiv P_2^\sharp$.
2. If $P_1 \to_{\mathbf{1et}} P_2$, then $P_1^\sharp \to_\beta P_2^\sharp$.
3. If $P_1 \to_{\mathbf{1et}_\eta} P_2$, then $P_1^\sharp \to_\eta P_2^\sharp$. □

6 Residuated CPS-translation

As expected from the previous results, the CPS-translation forms a residuated mapping from $\lambda 2$ to *Univ*.

Theorem 1 (Residuated CPS-trans.) *The CPS-translation $*$ is a residuated mapping from $\Lambda 2$ to Univ.*

Proof. From Proposition 2, Lemmata 1 and 4, and Corollary 1, the translation $*$ is a residuated mapping. In other words, for any $P \in \mathit{Univ}$, we have

$$\{M \in \Lambda 2 \mid M^* \sqsubseteq P\} = \downarrow P^\sharp.$$

In fact, from Lemma 1 and Corollary 1, we have $\downarrow P^\sharp \subseteq \{M \in \Lambda 2 \mid M^* \sqsubseteq P\}$. On the other hand, from Lemma 4 and Corollary 1, the inverse direction $\{M \in \Lambda 2 \mid M^* \sqsubseteq P\} \subseteq \downarrow P^\sharp$ holds true. □

We summarize results induced from the discussion above.

Corollary 2 *1. $\lambda 2$ is strongly normalizing if and only if Univ is strongly normalizing.*

2. $\lambda 2$ is weakly normalizing if and only if Univ is weakly normalizing.

3. $\lambda 2$ is Church-Rosser if and only if Univ is Church-Rosser.
We remark that $\Lambda^\exists$ itself is not Church-Rosser.

4. Let $\downarrow P$ be $\{Q \mid P \twoheadrightarrow_{\lambda^\exists} Q\}$ for $P \in$ Univ. Then the inverse image under $*$ of $\downarrow P$ is a principal down-set generated by $P^\sharp \in \Lambda 2$.

5. Given the CPS-translation $*$. Then an existence of its residual (inverse translation) is unique.

6. Define $P_1 \sim_{\beta\eta} P_2$ by $\Downarrow_{\beta\eta} P_1 \equiv \Downarrow_{\beta\eta} P_2$ for $P_1, P_2 \in$ Univ. There exists a bijection $\star$ between $\Lambda 2$ and Univ/ $\sim_{\beta\eta}$. In particular, there exists a one-to-one correspondence between $\lambda 2$-normal forms and Univ-normal forms.

7. Let $\downarrow_{\lambda^\exists}[\Lambda 2]^*$ be the down-set generated by $[\Lambda 2]^*$, i.e., $\{P \mid M^* \twoheadrightarrow_{\lambda^\exists} P$ for some $M \in \Lambda 2\}$. Let $\uparrow_{\beta\eta}[\Lambda 2]^*$ be the up-set generated by $[\Lambda 2]^*$, i.e., $\{P \in$ Univ $\mid P \twoheadrightarrow_{\beta\eta} M^*$ for some $M \in \Lambda 2\}$.
Then we have $\downarrow_{\lambda^\exists}[\Lambda 2]^* \subseteq$ Univ $= \uparrow_{\beta\eta}[\Lambda 2]^*$. We remark that $\subseteq$ is strict. For instance, $xa \in \downarrow_{\lambda^\exists}[\Lambda 2]^*$ and $(\lambda a.xa)a \in$ Univ, but $(\lambda a.xa)a \not\in \downarrow_{\lambda^\exists}[\Lambda 2]^*$.

Proof.

1. If $M_1 \rightarrow_{\lambda 2} M_2$, then we have $M_1^* \rightarrow_{\lambda\exists}^{+} M_2^*$ by induction on the derivation. Therefore, strong normalization of *Univ* implies that of $\lambda 2$.

   On the other hand, $\rightarrow_{\beta\eta}$ in *Univ* is strongly normalizing. If *Univ* has an infinite reduction path of $\rightarrow_{\lambda\exists}$, then the path should contain an infinite reduction path consisting of $\rightarrow_{1\bullet t, 1\bullet t_\eta}$. Now, from Lemma 4, $\lambda 2$ has an infinite reduction path of $\rightarrow_{\beta\eta}$. Hence, strong normalization of $\lambda 2$ implies that of *Univ*.

2. From the monotone translations between $\Lambda 2$ and *Unvi*, and the one-to-one correspondence between $\lambda 2$-normal forms and *Univ*-normal forms.

3. $\Lambda 2$ and *Univ* form the so-called Galois connection under $*$ and $\sharp$.

4. The CPS-translation $*$ forms a residuated mapping.

5. Suppose we had two inverse translations $\sharp_1$ and $\sharp_2$, then $P^{\sharp_1} \equiv P^{\sharp_2}$ for any $P \in Univ$. Because we have $P \twoheadrightarrow_{\beta\eta} P^{\sharp_1 *}$ for any $P \in Univ$ from Corollary 1 (1). Hence, we have $P^{\sharp_2} \equiv (P^{\sharp_1 *})^{\sharp_2} \equiv P^{\sharp_1}$ from Lemma 4 (1).

6. Since $\sim_{\beta\eta}$ is an equivalence relation over *Univ*, we take
   $[P]_{\sim_{\beta\eta}} = \{P' \in Univ \mid P \sim_{\beta\eta} P'\}$ for $P \in Univ$.
   Then we define $\star(M) = [M^*]_{\sim_{\beta\eta}}$. In other words,
   $\star(M) = \uparrow_{\beta\eta}(M^*) = \{P \in Univ \mid P \twoheadrightarrow_{\beta\eta} M^*\}$.
   Then $\star : \Lambda 2 \rightarrow Univ/\sim_{\beta\eta}$ is a bijection. In fact, for any $[P] \in Univ/\sim_{\beta\eta}$, there exists $M \in \Lambda 2$ such that $\star(M) = [P]$. Because we take $M \equiv P^\sharp$, whence $P \twoheadrightarrow_{\beta\eta} (P^\sharp)^*$ and $\star(P^\sharp) = [P]$. On the other hand, suppose $M_1 \not\equiv M_2$. Then $\star(M_1) \neq \star(M_2)$, since M_1^* and M_2^* are distinct $\beta\eta$-normal forms.

7. For any $M \in \Lambda 2$, we have $M^* \in Unvi$, and *Univ* is a down-set with respect to $\twoheadrightarrow_{\lambda\exists}$. Then we have $\downarrow_{\lambda\exists}[\Lambda 2]^* \subseteq Univ$.

   For any $P \in Univ$, we have $P^\sharp \in \Lambda 2$ and $P \twoheadrightarrow_{\beta\eta} P^{\sharp *}$ from Lemma 1. Hence, $P \in \uparrow_{\beta\eta}[\Lambda 2]^*$ holds true. The inverse direction is clear, and therefore we have $Univ = \uparrow_{\beta\eta}[\Lambda 2]^*$. □

It is remarked that instead of pseudo-terms, when we take account of well-typed terms, the binary relations $\twoheadrightarrow_{\lambda 2}$ and $\twoheadrightarrow_{\lambda\exists}$ form partial orders on λ-terms.

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